Statistical mechanics of paths with curvature-dependent action

1988 J. Phys. A: Math. Gen. 21 981

(http://iopscience.iop.org/0305-4470/21/4/025)

J. Phys. A: Math. Gen. **21** (1988) 981-1000. Printed in the UK

Statistical mechanics of paths with curvature-dependent action

Jan Ambjørn†, Bergfinnur Durhuus‡ and Thordur Jonsson§

† The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
‡ The Mathematics Institute, Universitetsparken 5, 2100 Copenhagen Ø, Denmark
§ NORDITA, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark

Received 29 April 1987

Abstract. We analyse the scaling limit of discretised random paths with curvature-dependent action. For finite values of the curvature coupling constant the theory belongs to the universality class of simple random walks. It is possible to define a non-trivial scaling limit if the curvature coupling tends to infinity. We compute exactly the two-point function in this limit and discuss the relevance of our results for random surfaces and string theories.

1. Introduction

The theory of random walks is important in many branches of physics. In this paper we study a new class of random walks, where the action, or the transition function, depends on the curvature. The theory of random walks with curvature-dependent action has a long history in polymer physics [1, 2] and we have used the term 'new' above because we are interested in a particular scaling limit of the theory which turns out to be different from the scaling limit of the ordinary random walk.

An important motivation for studying the theory of random walks with curvature-dependent action stems from string theory. String theory is most naturally formulated as a first quantised theory, i.e. a theory of random surfaces, and, as emphasised by Polyakov [3], extrinsic curvature is likely to play an important role. The same conclusions were reached during the study of discretised random surface models which could serve as rigorous regularisations of string theories [4-6]. The major problem in these regularised random surface models was the non-vanishing of the string tension at the critical point [4, 7]. In order to obtain a random surface theory with mass and string tension vanishing at the same critical point it seems that one must have at least two independent coupling constants that can be tuned simultaneously and there are strong arguments in favour of an extrinsic curvature coupling as one of them [4, 6] (see also [8]). It should also be mentioned that an extrinsic curvature term arises in the action of stiff membranes [9].

Contrary to the theory of random surfaces with curvature-dependent action, the corresponding theory of random paths can be completely solved, as we will show below. It is noteworthy that the qualitative aspects of the theory are exactly what one hopes for in the theory of random surfaces.

We now give a quick overview of our results. Let $x(s)$ be a smooth path in R^d. The curvature of x at s is given by

$$k=\frac{[\dot{x}^2\ddot{x}^2-(\dot{x}\cdot\ddot{x})^2]^{1/2}}{\dot{x}^2} \tag{1.1}$$

where the dot denotes differentiation with respect to the parameter s. The curvature-dependent action of paths in the continuum is given by

$$S = \beta \int |\dot{x}|^{\alpha_1} \, \mathrm{d}s + \lambda \int k^{\alpha_2} \, \mathrm{d}s \tag{1.2}$$

where α_1 and α_2 are positive real numbers. The functional integral corresponding to S has been studied directly in the continuum by saddle point methods for large d [10, 11]. A related problem is considered in [12].

We shall study the two most natural discretised versions of (1.2), i.e. random walks in a hypercubic lattice and piecewise straight paths (random flight model). Our results are of two kinds.

(i) Finite values of the bare coupling constant λ (i.e. the value of λ used in the discretised models) do not change the statistical behaviour of the paths. For finite bare λ we have the same critical behaviour as for ordinary simple random walks, i.e. the continuum theory defined at the critical point (β_c, λ_c), $\lambda_c < \infty$, is identical to the one corresponding to (1.2) with $\lambda = 0$.

(ii) When the bare coupling λ tends to infinity, the critical behaviour of paths changes qualitatively. Tangent vectors to paths develop long-range correlations and the two-point function has an anomalous dimension $\eta = 1$.

For the lattice model we compute in this case the scaling limit of the two-point function in a closed form in § 2. This two-point function only has the symmetry of the hypercubic lattice, so special care has to be taken in order to get a rotationally invariant theory.

The random flight model is considered in § 3. It is manifestly Euclidean invariant. For this model the long-range correlation of tangents is (in the scaling limit) described by diffusion on the unit sphere. In the lattice case the correlations of tangents can be described by diffusion in a finite set, i.e. the intersection of the unit sphere in $\mathbb{R}^d$ with the coordinate axes. This is the reason for the special attention which one has to pay to rotational invariance in the lattice case.

In § 3 we also compare our results with a class of random walks for which the curvature-dependent term in the action is not scale invariant. The continuum limit of these walks can be constructed and they all have anomalous dimension $\eta = 1$. One of these limits is the Ornstein-Uhlenbeck velocity process.

For the discretised random walk models we study here, the steps are not independent random variables. In the appendix we show that the generalisation of the central limit theorem to the case of uniformly mixing random variables can be applied to prove that for any fixed finite λ the endpoint distribution for the walks is asymptotically Gaussian.

Finally, in the conclusion, we compare the different scaling limits that we construct and discuss the relevance of our results for random surfaces and string theories.

2. Random walk in $\mathbb{Z}^d$

In this section we consider random walks in the lattice $\mathbb{Z}^d$. When the action of the walks is taken to be the natural lattice version of (1.2), we compute the Fourier transform of the scaling limit of the two-point function in a closed form and discuss its properties.

Let $e_1, \ldots, e_d$ be the standard orthonormal basis for $\mathbb{Z}^d$. Define the unit vectors f_i^α by $f_i^\alpha = (-1)^\alpha e_i$, where $\alpha = 0, 1$. A random walk ω in $\mathbb{Z}^d$ of length $|\omega| = n$ is a sequence of $n+1$ points $x_0, x_1, \ldots, x_n \in \mathbb{Z}^d$ such that for $j = 0, \ldots, n-1$, $x_{j+1} - x_j$ equals one of the unit vectors f_i^α.

We define an energy functional, $A(\omega)$, for all random walks of finite length by the formula

$$A(\omega) = \sum_{j=2}^{n} a(x_j - x_{j-1}, x_{j-1} - x_{j-2}) \tag{2.1}$$

where $a(\cdot, \cdot)$ is a positive function which gives the action of individual steps in the walk. For an isotropic walk, $a(x, y)$ is only a function of the inner product $x \cdot y$. This is the case we shall be concerned with even though the general case might not be devoid of interest and our formalism can be applied to it.

Our goal is to analyse the statistical mechanics of random walks with Boltzmann factors $\exp(-A(\omega))$. The lattice discretisation of (1.2) gives

$$a(x, y) = \begin{cases} \beta & \text{if } \quad x \cdot y = 1 \\ \beta + 2\lambda & \text{if } \quad x \cdot y = -1 \\ \beta + \sqrt{2}\lambda & \text{if } \quad x \cdot y = 0. \end{cases} \tag{2.2}$$

This is the case studied in [13]. It is convenient to introduce the notation

$$\exp[-a(f_i^\alpha, f_j^\beta)] = t_{ij}^{\alpha\beta} \tag{2.3}$$

where

$$t_{ij}^{\alpha\beta} \equiv \begin{cases} t_1 & \text{if } \quad i = j,\ \alpha = \beta \\ t_2 & \text{if } \quad i = j,\ \alpha \neq \beta \\ t_3 & \text{if } \quad i \neq j. \end{cases} \tag{2.4}$$

The two-point function, $G(x, y)$, is defined by

$$G(x, y) = \sum_{\omega : x \to y} \exp(-A(\omega)) \tag{2.5}$$

where the sum is over all walks ω from x to y. It is clear that $G(x, y)$ only depends on $x - y$ so we shall use the notation $G(x, y) = G(x - y)$. The susceptibility χ is defined by

$$\chi(\beta) = \sum_x G(x) \tag{2.6}$$

and equals $\xi(1-\xi)^{-1}$, where

$$\xi = t_1 + t_2 + 2(d-1)t_3. \tag{2.7}$$

The function $G(x)$ is divergent for $\xi > 1$ but finite for $\xi \leqslant 1$. When we choose coupling constants on which ξ depends via the t_i, we refer to the surface $\xi = 1$ in the coupling constant space as the critical surface.

We introduce the normalised expectation for random walk observables by

$$\langle\langle \cdot \rangle\rangle = \chi^{-1} \sum_{x \in \mathbb{Z}^d} (\cdot) G(x). \tag{2.8}$$

The correlation function of tangents to the walks is easily calculated [13]:

$$\begin{aligned} g(n) &\equiv \langle\langle (x_1 - x_0) \cdot (x_n - x_{n-1}) \rangle\rangle \\ &= \exp(-mn) \end{aligned} \tag{2.9}$$

where

$$m = -\ln\left(\frac{t_1 - t_2}{\xi}\right). \tag{2.10}$$

There are two mass scales in the theory, namely m, given by (2.10), and μ, the mass appearing in the exponential decay of $G(x)$ as x goes to infinity along a coordinate axis. Roughly speaking, μ is a measure of how long the typical walks are and m of how straight they are. The mass μ vanishes at any point on the critical surface, whereas m vanishes when $t_2 = t_3 = 0$. Thus, the scaling limit of the tangent-tangent correlation function can only exist at $t_2 = t_3 = 0$.

For the computation of $G(x)$ we need to introduce some auxiliary quantities. Let us define

$$G_n^{i\alpha j\beta}(x) = \sum_{\substack{\omega : 0 \to x, |\omega| = n \\ x_1 - x_0 = f_i^\alpha, x_n - x_{n-1} = f_j^\beta}} \exp(-A(\omega)). \tag{2.11}$$

We have the recursion relation

$$G_{n+1}^{i\alpha j\beta}(x) = \sum_{\gamma=1}^{2} \sum_{k=1}^{d} G_n^{i\alpha k\gamma}(x - f_j^\beta) t_{kj}^{\gamma\beta}. \tag{2.12}$$

Fourier transforming (2.12) we find

$$\tilde{G}_{n+1}^{i\alpha j\beta}(q) = \sum_{\gamma=1}^{2} \sum_{k=1}^{d} \tilde{G}_n^{i\alpha k\gamma}(q) t_{kj}^{\gamma\beta} \exp(iq \cdot f_j^\beta) \tag{2.13}$$

where

$$\tilde{G}_n^{i\alpha j\beta}(q) = \sum_{x \in \mathbb{Z}^d} G_n^{i\alpha j\beta}(x) \exp(-iq \cdot x). \tag{2.14}$$

It is convenient to define a $2d \times 2d$ matrix $\mathbb{M}(q)$ by

$$\mathbb{M}(q)_{ij}^{\alpha\beta} = t_{ij}^{\alpha\beta} \exp(-iq \cdot f_i^\alpha). \tag{2.15}$$

Similarly, we define the matrices $\mathbb{G}_n(x)$, $\mathbb{G}(x)$, $\tilde{\mathbb{G}}_n(q)$, $\tilde{\mathbb{G}}(q)$ by

$$\mathbb{G}_n(x)_{ij}^{\alpha\beta} = G_n^{i\alpha j\beta}(x) \tag{2.16}$$

$$\mathbb{G}(x) = \textstyle\sum_{n=1}^{\infty} \mathbb{G}_n(x) \tag{2.17}$$

$$\tilde{\mathbb{G}}_n(q)_{ij}^{\alpha\beta} = \tilde{G}_n^{i\alpha j\beta}(q) \tag{2.18}$$

$$\tilde{\mathbb{G}}(q) = \sum_{n=1}^{\infty} \tilde{\mathbb{G}}_n(q). \tag{2.19}$$

Equation (2.13) implies that

$$\tilde{\mathbb{G}}_{n+1}(q) = \mathbb{M}(q)^n \tilde{\mathbb{G}}_1(q) \tag{2.20}$$

and hence

$$\tilde{\mathbb{G}}(q) = (\mathbb{1} - \mathbb{M}(q))^{-1} \tilde{\mathbb{G}}_1(q). \tag{2.21}$$

The existence of $(1 - \mathbb{M}(q))^{-1}$ follows from the convergence of the sum (2.19) for any $q \in \mathbb{R}^d$, provided $\xi < 1$. We therefore have

$$\mathbb{G}(x) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \mathrm{d}q \exp(iq \cdot x)(\mathbb{1} - \mathbb{M}(q))^{-1} \tilde{\mathbb{G}}_1(q) \tag{2.22}$$

which leads to a formula for the two-point function, since

$$G(x)=\sum_{i,j,\alpha,\beta}\mathbb{G}(x)_{ij}^{\alpha\beta}. \tag{2.23}$$

Note that

$$G_1^{i\alpha j\beta}(x)=\delta_{ij}\delta_{\alpha\beta}\delta_{xf_i^\alpha} \tag{2.24}$$

so

$$\tilde{G}_1^{i\alpha j\beta}(q)=\delta_{ij}\delta_{\alpha\beta}\exp(-\mathrm{i}q\cdot f_i^\alpha). \tag{2.25}$$

From (2.22)-(2.25) one can derive a closed formula for $G(x)$. We prefer to bypass this formula in deriving the salient features of $G(x)$ and instead write down explicit expressions for the scaling limits of $G(x)$.

In order to calculate μ, the mass governing the exponential decay of $G(x)$, it suffices to find the pole of the matrix elements of $\tilde{G}(q)$ which is closest to the real q_k axis, when $q_j=0$ for $j\neq k$. These poles are located exactly where the inverse of $\mathbb{1}-\mathbb{M}(q)$ does not exist, i.e. where $\det(\mathbb{1}-\mathbb{M}(q))=0$.

The matrix $\mathbb{M}(q)$ has a simple block structure, so this determinant is easily calculated and equals

$$\left(1+2t_3\sum_{j=1}^{d}\frac{(t_1-t_2-\cos q_j)}{h(q_j)}\right)\prod_{i=1}^{d}h(q_i) \tag{2.26}$$

where

$$h(q_i)=1+(t_1-t_3)^2+(t_2-t_3)^2-2(t_1-t_3)\cos q_i. \tag{2.27}$$

For $0<1-\xi\ll 1$ and $t_1\neq 1$ one can calculate from (2.26) and (2.27) that the determinant is zero for

$$\begin{aligned} q_k&=\pm\mathrm{i}\frac{1-\delta}{[t_3(1+\delta)]^{1/2}}(1-\xi)^{1/2}+\mathrm{O}(1-\xi)\\ q_j&=0 \qquad j\neq k \end{aligned} \tag{2.28}$$

where $\delta\equiv t_1-t_3\neq 1$ and we have for simplicity put $t_2=t_3$. If we define the critical exponent of μ, denoted by ν, by the formula

$$\mu\sim(1-\xi)^{\nu} \tag{2.29}$$

as $\xi\uparrow 1$, we conclude from (2.28) that $\nu=\frac{1}{2}$ except at the point $t_1=1$, $t_2=t_3=0$, where a more detailed analysis is required. It is easy to extend the argument above to show that, for $\xi<1$, the power corrections to the exponential decay of $G(x)$ are of Ornstein-Zernike type.

We now turn our attention to the scaling limit. A scaling limit of $G(x)$ can be constructed at any point on the critical surface. For the sake of simplicity we take $t_2=t_3$ as above and write

$$\begin{aligned} t_1&=\exp(-\beta)\\ t_2&=\exp(-\beta-\lambda). \end{aligned} \tag{2.30}$$

The region in the (β,λ) plane, where $\xi<1$, has the shape indicated in figure 1. We choose a path, $\theta\to(\lambda(\theta),\beta(\theta))$, in the coupling constant space such that

$$\theta\mu(\beta(\theta),\lambda(\theta))\to\mu_*>0$$

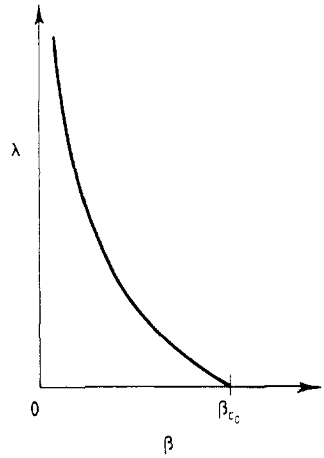

Figure 1. The critical curve $\lambda = \lambda_c(\beta)$ in the (β, λ) coupling constant plane. The curve starts at the critical point $(\beta, \lambda) = (\beta_{c_0}, 0)$ corresponding to the ordinary random walk.

as $\theta \to \infty$. Given $y \in \mathbb{R}^d$ we let θ tend to infinity along a discrete sequence such that $\theta y \in \mathbb{Z}^d$. The scaling limit of $G(x)$ is defined by

$$G^*(y) = \lim_{\theta \to \infty} \theta^{d-2+\eta} G(\theta y) \tag{2.31}$$

if the limit exists. The exponent η in (2.31) is the anomalous dimension of the two-point function.

Approaching the critical surface at some point different from $t_1 = 1$, one can check, by a similar computation as the one that led to (2.28), that $\det(\mathbb{1} - \mathbb{M}(q))$ tends to 0 as $1 - \xi$ at $q = 0$. By a short computation we then find

$$\theta^{-2}(\mathbb{1} - \mathbb{M}(\theta^{-1}q))^{-1} \to (q^2 + \mu_*^2)^{-1}\mathbb{M}^* \tag{2.32}$$

as $\theta \to \infty$, where $\mathbb{M}^*$ is a calculable constant matrix. It follows now from (2.22), (2.23) and (2.31) that

$$G^*(x) = \text{constant}(-\Delta + \mu_*^2)^{-1}(0, x) \tag{2.33}$$

where Δ is the Laplacian on $\mathbb{R}^d$, i.e. the scaling limit is the usual Brownian motion.

We now turn our attention to the more interesting scaling limit at $t_1 = 1$, $t_2 = t_3 = 0$, i.e. $\beta = 0$ and $\lambda = \infty$. We require the scaling limits of the masses m and μ both to exist and be finite, i.e.

$$\begin{aligned} &\lim_{\theta \to \infty} \theta^{-1} \mu(\beta(\theta), \lambda(\theta)) = \mu_* > 0 \\ &\lim_{\theta \to \infty} \theta^{-1} m(\beta(\theta), \lambda(\theta)) = m_* > 0. \end{aligned} \tag{2.34}$$

It follows from (2.10) that

$$2d \exp(-\lambda(\theta)) = m_*/\theta \tag{2.35}$$

and from (2.28) one can see that we must let

$$1 - \exp(-\beta(\theta)) = b\theta^{-1} \tag{2.36}$$

for some positive constant b. Now we are prepared to let $\theta\to\infty$. We obtain

$$\lim_{\theta\to\infty}\theta(\mathbb{1}-\mathbb{M}(q\theta^{-1}))=\mathbb{A}(q)$$

$$\equiv\begin{pmatrix} b+\mathrm{i}q_1 & -m & & \cdots & & -m \\ -m & b-\mathrm{i}q_1 & -m & & & \\ -m & -m & b+\mathrm{i}q_2 & & & \vdots \\ \vdots & \vdots & & \ddots & b+\mathrm{i}q_d & -m \\ -m & & \cdots & & -m & b-\mathrm{i}q_d \end{pmatrix} \tag{2.37}$$

where we use the notation $m=m^*/2d$. It is straightforward to compute the inverse of $\mathbb{A}$ and from (2.20), (2.21) and (2.31) we obtain

$$\tilde{G}^*(q)=\sum_{i=1}^{d}\frac{2M}{M^2+q_i^2}\left(1-\sum_{i=1}^{d}\frac{2mM}{M^2+q_i^2}\right)^{-1} \tag{2.38}$$

where

$$M=b+m.$$

The above calculation is easily generalised to the case $t_2\neq t_3$, but that case has no new features.

We have now demonstrated explicitly that $\eta=1$. If we exhibit the dependence of the two-point function on the renormalised 'masses' M and m by writing $\tilde{G}^*(q;M,m)$ and $G^*(x;M,m)$ instead of $\tilde{G}^*(q)$ and $G^*(x)$, then $G^*(\theta x;M,m)=\theta^{-d+1}G^*(x;\theta M,\theta m)$. This is another manifestation of the anomalous dimension $\eta=1$. In [13] we used Fisher's scaling relation to prove this fact, so now we have proven Fisher's scaling relation for the random walk model at $\lambda=\infty$, $\beta=0$.

The formula (2.38) has several noteworthy features. First of all, it is *not rotationally invariant* and only has the symmetry of the original lattice. This is not surprising because we are taking the scaling limit with the requirement that the correlation length of tangents to the walk is finite and thus the continuum walks remember the lattice structure since the tangents can only point in coordinate directions. The only place on the critical surface where this can happen is $(\beta,\lambda)=(0,\infty)$ because at other points the correlation length of tangents is zero.

From the condition $\xi<1$ it follows that $b>(2d-1)m$, i.e.

$$M>2dm=m_*. \tag{2.39}$$

Hence

$$D(q)\equiv1-\sum_{i=1}^{d}\frac{2mM}{M^2+q_i^2}>0 \tag{2.40}$$

for all $q\in\mathbb{R}^d$. The mass μ_* is given by the zeros of the function $f(z)=D((z,0,\ldots,0))$ closest to the real axis, so

$$\mu_*=M\left(\frac{M-2dm}{M-2(d-1)m}\right)^{1/2}<M. \tag{2.41}$$

The mass that governs the exponential decay of $G^*(x)$ in directions different from those of the coordinate axes is larger than μ_*, as can be seen from the structure of the zero set of $D(q)$. However, with a suitable anisotropic rescaling of x space, one could obtain the same exponential decay in all directions and then $G(x)$ would be approximately rotationally invariant at large distances. This corresponds simply to a choice of a norm, different from the Euclidean one, to define distances in x space.

Of course, one can construct a rotationally invariant two-point function by averaging over the rotation group. In two dimensions it is easy to carry out the computation and one obtains

$$\tilde{G}^*(q) \equiv \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \tilde{G}^*(|q|\cos\theta, |q|\sin\theta)$$
$$= \frac{\frac{1}{2}M^2 + \frac{1}{4}q^2}{\{q^6(1-2m/M) + q^4[M\mu_* + 4(M-2m)^2] + 8q^2M^2\mu_*(M-2m) + 4M^4\mu_*^2\}^{1/2}}. \tag{2.42}$$

In the massless case, $\mu_* = 0$, this takes the simple form

$$\frac{1}{2\sqrt{2}} \frac{1}{q^2} (2M^2 + q^2)^{1/2}. \tag{2.43}$$

In the next section and the conclusion we compare the above result with the rotationally invariant two-point function constructed in the next section by Euclidean invariant regularisation.

3. Piecewise linear random walks

In this section we consider a Euclidean invariant random walk with curvature-dependent action. It is defined by the unnormalised transition amplitude

$$\tilde{P}(r_1, r_2) = \exp[-\beta h(|r_2|) - \lambda f(\theta(r_1, r_2))] \tag{3.1}$$

where $\theta(r_1, r_2)$ is the angle between the two subsequent steps r_1 and r_2 in $\mathbb{R}^d$, h is an arbitrary non-negative function on the positive real line with suitable growth properties and f is a non-negative continuous function on $[0, \pi]$ such that $f(0) = 0$ and $f(\theta) \neq 0$ if $\theta \neq 0$.

Thus, the two-point function is given by

$$G_{\beta,\lambda}(0, x) = \sum_{n=1}^{\infty} \int \mathrm{d}r_1 \ldots \mathrm{d}r_{n-1}\, P_0(r_1)\tilde{P}(r_1, r_2) \ldots \tilde{P}(r_{n-1}, x - r_1 - r_2 - \ldots - r_{n-1}) \tag{3.2}$$

where P_0 is the distribution of the first step, which may be chosen arbitrarily (however, see the appendix).

For $\lambda = 0$ the central limit theorem for sums of independent random variables implies that the continuum limit of $G_{\beta,0}$ is the free propagator, corresponding to Brownian motion. For any fixed $\lambda > 0$ we prove in the appendix that the same conclusion holds by exploiting a central limit theorem for weakly dependent random variables.

Noticing the scale invariance of the second term in the exponent in (3.1), it follows that the normalised transition probability $P(r_1, r_2)$ in (3.1) can be rewritten in the factorised form

$$P(r_1, r_2) = \frac{1}{N_1(\beta)} \exp[-\beta h(|r_2|)] \frac{1}{N_2(\lambda)} \exp[-\lambda f(\theta(r_1, r_2))] \tag{3.3}$$

where

$$N_1(\beta)=\int_0^\infty \exp[-h(|r|)]|r|^{d-1}\,\mathrm{d}|r| \tag{3.4}$$

$$N_2(\lambda)=\int_{S^{d-1}} \exp[-\lambda f(\theta(\hat{r}_1,\hat{r}_2))]\,\mathrm{d}\Omega(\hat{r}_2) \tag{3.5}$$

$\hat{r}=r/|r|$ and $\mathrm{d}\Omega(\hat{r})$ is the uniform measure on S^{d-1} such that $\mathrm{d}^d r=|r|^{d-1}\,\mathrm{d}|r|\,\mathrm{d}\Omega(\hat{r})$.

Thus, expectation values of functions depending only on the normalised step vectors $\hat{r}_i$ with respect to the measure defined by P are equal to the expectation values (to be denoted by $\langle\cdot\rangle_\lambda$) with respect to the measure defined by the transition probabilities

$$K_\lambda(\hat{r}_1,\hat{r}_2)=\frac{1}{N_2(\lambda)}\exp[-\lambda f(\theta(\hat{r}_1,\hat{r}_2))] \tag{3.6}$$

if P_0 is chosen to depend on $|r_1|$ only, i.e. P_0 is rotationally invariant, which we shall assume in the following.

In particular, the tangent-tangent correlation between the directions of the first and the nth step is given by

$$\langle\hat{r}_1\cdot\hat{r}_n\rangle_\lambda=\int \hat{r}_1\cdot\hat{r}_n K_\lambda(\hat{r}_1,\hat{r}_2)\ldots K_\lambda(\hat{r}_{n-1},\hat{r}_n)\,\mathrm{d}\Omega(\hat{r}_1)\ldots\mathrm{d}\Omega(\hat{r}_n). \tag{3.7}$$

More generally, expectation values of functions of the form $f_1(\hat{r}_1)\ldots f_n(\hat{r}_n)$ are given by

$$\langle f_1(\hat{r}_1)\ldots f_n(\hat{r}_n)\rangle_\lambda=\int f_1(\hat{r}_1)\ldots f_n(\hat{r}_n)K_\lambda(\hat{r}_1,\hat{r}_2)\ldots K_\lambda(\hat{r}_{n-1},\hat{r}_n)\,\mathrm{d}\Omega(r_1)\ldots\mathrm{d}\Omega(r_n) \tag{3.8}$$

which yields an interpretation of the model, as far as these expectation values are concerned, as a one-dimensional classical spin chain with transfer matrix K_λ.

The kernel $K_\lambda(\cdot,\cdot)$ defines a bounded operator K_λ on the Hilbert space $\mathscr{H}=L^2(S^{d-1})$ of square-integrable functions on S^{d-1} (with respect to $\mathrm{d}\Omega$) by

$$(K_\lambda\varphi)(\hat{r}_1)=\int K_\lambda(\hat{r}_1,\hat{r}_2)\varphi(\hat{r}_2)\,\mathrm{d}\Omega(\hat{r}_2). \tag{3.9}$$

The operator K_λ has norm 1. 1 is a non-degenerate eigenvalue with eigenfunction 1 and -1 is not an eigenvalue (see the appendix). For $a\in\mathbb{R}^d\backslash\{0\}$ the function

$$\varphi_a(\hat{r})=a\cdot\hat{r} \tag{3.10}$$

is an eigenfunction of K_λ, since

$$(K_\lambda\varphi_a)(\hat{r}_1)=\frac{1}{N_2(\lambda)}\int \exp[-\lambda f(\theta(\hat{r}_1,\hat{r}_2))]a\cdot\hat{r}_2\,\mathrm{d}\Omega(\hat{r}_2) \tag{3.11}$$

considered as a function of a and $\hat{r}$ is invariant under simultaneous rotations of a and $\hat{r}_1$ (since $\mathrm{d}\Omega$ is rotation invariant) and is linear in a. Denoting the eigenvalue of φ_a by $\alpha(\lambda)$, we obtain from (3.7)

$$\langle\hat{r}_1\cdot\hat{r}_n\rangle=\alpha(\lambda)^n \qquad n\in\mathbb{N} \tag{3.12}$$

and $|\alpha(\lambda)|<1$ for $0\leqslant\lambda<\infty$, according to the remarks above. Thus the correlation length $\xi(\lambda)=(-\log|\alpha(\lambda)|)^{-1}$ only diverges if $|\alpha(\lambda)|\to 1$. We shall now show that

$\alpha(\lambda)\to 1$, if $\lambda\to\infty$, and that the corresponding scaling limit of the expectation values (3.8) is given by Brownian motion on the sphere S^{d-1}.

We start by proving that there exists a function $\lambda(n)$, $n\in\mathbb{N}$, such that

$$\lim_{n\to\infty} K_{\lambda(n)}^{tn}\varphi = \exp(-tL)\varphi \qquad \forall\varphi\in\mathscr{H},\ t\geqslant 0 \tag{3.13}$$

where L is the Laplace-Beltrami operator on the sphere S^{d-1}.

Since K_λ is a rotationally invariant operator, i.e. $K_\lambda(O\hat{r}_1, O\hat{r}_2)=K_\lambda(\hat{r}_1,\hat{r}_2)$ for any orthogonal transformation $O:\mathbb{R}^d\to\mathbb{R}^d$, it follows that K_λ commutes with the orthogonal group acting on $\mathscr{H}$ by $f\to f\circ O$. Hence K_λ has the same eigenfunctions as L with the same multiplicity of eigenvalues. To be more precise, denoting by α_l, $l=0,1,2,\ldots$, the eigenvalues of L in increasing order and by d_l the multiplicity of α_l, we may choose an orthonormal basis of eigenfunctions $\varphi_{l,i}$, $i=1,\ldots,d_l$, $l=0,1,2,\ldots$. These functions are also eigenfunctions of K_λ and the eigenvalues of $\varphi_{l,1},\ldots,\varphi_{l,d_l}$ with respect to K_λ are equal. We shall denote this eigenvalue by $\beta_l(\lambda)$.

To prove (3.13) it is enough to prove that we can choose $\lambda(n)$ such that

$$\lim_{n\to\infty}\beta_l(\lambda(n))^n=\exp(-\alpha_l) \qquad \forall l=0,1,2,\ldots \tag{3.14}$$

since this implies that, for $t\geqslant 0$,

$$\lim_{n\to\infty}(\varphi, K_{\lambda(n)}^{tn}\psi)=(\varphi,\exp(-tL)\psi) \tag{3.15}$$

for all φ, $\psi\in\mathscr{H}$ which are finite linear combinations of the $\varphi_{l,i}$ (here $(\cdot,\cdot)$ denotes the inner product in $\mathscr{H}$). Since $\|K_\lambda\|=\|\exp(-L)\|=1$ and the set of finite linear combinations of $\varphi_{l,i}$ is dense in $\mathscr{H}$, it follows that (3.15) holds for all φ, $\psi\in\mathscr{H}$. From this (3.13) follows for all $\varphi\in\mathscr{H}$, since

$$\begin{aligned}\|K_{\lambda(n)}^{tn}\varphi-\exp(-tL)\varphi\|^2 &= (\varphi, K_{\lambda(n)}^{2tn}\varphi)-(K_{\lambda(n)}^{tn}\varphi,\exp(-tL)\varphi)-(\exp(-tL)\varphi, K_{\lambda(n)}^{tn}\varphi)\\ &\quad +(\varphi,\exp(-2tL)\varphi)\to 0 \qquad \text{as} \qquad n\to\infty.\end{aligned}$$

In order to prove (3.14) we fix a North pole $\hat{n}$ on S^{d-1} and denote the azimuthal angle of $\hat{r}\in S^{d-1}$ with respect to $\hat{n}$ by $\theta(\hat{r})$, i.e. $\cos\theta(\hat{r})=\hat{n}\cdot\hat{r}$. It is well known (see [14]) that it is possible to choose the functions $\varphi_{l,i}$ such that $\varphi_{l,1}$, $l=0,1,2,\ldots$, depends only on $\hat{r}$ through $\theta(\hat{r})$. We suppose in the following that we have made such a choice and denote the function $\varphi_{l,1}(\hat{r})$ as a function of θ by $\varphi_l(\theta)$. Inserting $\hat{r}_1=\hat{n}$ into the eigenvalue equation

$$\frac{1}{N_2(\lambda)}\int\exp[-\lambda f(\theta(\hat{r}_1,\hat{r}_2))]\varphi_{l,1}(\hat{r}_2)\,\mathrm{d}\Omega(\hat{r}_2)=\beta_l(\lambda)\varphi_{l,1}(\hat{r}_1) \tag{3.16}$$

we obtain

$$\frac{1}{N_2(\lambda)}\int\exp[-\lambda f(\theta(\hat{r}))]\varphi_l(\theta(\hat{r}))\,\mathrm{d}\Omega(\hat{r})=\beta_l(\lambda)\varphi_l(0). \tag{3.17}$$

Furthermore, it is well known that (see, e.g., [14])

$$(L\varphi_l)(0)=-\varphi_l''(\theta)-(d-2)\cot\theta\varphi_l'(\theta)|_{\theta=0}=-(d-1)\varphi_l''(\theta) \tag{3.18}$$

and

$$\varphi_l'(0)=0. \tag{3.19}$$

Using the Taylor expansion

$$\varphi_l(\theta)=\varphi_l(0)+\theta^2\varphi_l''(0)+\theta^3\psi(\theta)$$

where $|\psi(\theta)|=O(1)$, together with (3.18) in (3.17), we obtain

$$\beta_l(\lambda)=1-\frac{\alpha_l}{(d-1)N_2(\lambda)}\int \exp[-\lambda f(\theta(\hat{r}))]\theta(\hat{r})^2\,d\Omega(\hat{r})$$
$$+\frac{1}{\varphi_l(0)}\frac{1}{N_2(\lambda)}\int \exp[-\lambda f(\theta(\hat{r}))]\theta(\hat{r})^3\psi(\theta(\hat{r}))\,d\Omega(\hat{r}). \tag{3.20}$$

Next we note that the quantity

$$c_\lambda=\frac{1}{(d-1)N_2(\lambda)}\int \exp[-\lambda f(\theta(\hat{r}))]\theta(\hat{r})^2\,d\Omega(\hat{r}) \tag{3.21}$$

tends to zero as $\lambda\to\infty$. This follows from the fact that f by assumption is continuous and only vanishes at $\theta=0$. Similarly, it follows that the last term in (3.20) tends to zero faster than c_λ as $\lambda\to\infty$. Thus, choosing $\lambda(n)$ such that

$$nc_{\lambda(n)}\to 1 \qquad \text{as} \qquad n\to\infty \tag{3.22}$$

we obtain

$$\lim_{n\to\infty}\beta_l(\lambda(n))^n=\lim_{n\to\infty}(1-\alpha_l c_{\lambda(n)}+o(c_{\lambda(n)}))^n=\exp(-\alpha_l) \tag{3.23}$$

as desired.

We remark that it follows from (3.13) that for any set $f_1,\ldots,f_k$ of bounded functions on S^{d-1} viewed as multiplication operators on $\mathscr{H}$ and for $t_1, t_2, \ldots, t_k \geqslant 0$

$$K^{t_1 n}_{\lambda(n)}f_1K^{t_2 n}_{\lambda(n)}f_2\ldots K^{t_k n}_{\lambda(n)}f_k\varphi\to\exp(-t_1L)f_1$$
$$\times\exp(-t_2L)f_2\ldots\exp(-t_kL)f_k\varphi \qquad \text{as} \qquad n\to\infty \tag{3.24}$$

for all $\varphi\in\mathscr{H}$.

Combining this result with (3.8) we obtain the continuum limit of the expectation value in (3.8) as

$$\lim_{n\to\infty}\langle f_1(\hat{r}_{t_1 n})\cdot f_2(\hat{r}_{(t_1+t_2)n})\cdot\ldots\cdot f_k(\hat{r}_{(t_1+\ldots+t_k)n})\rangle_{\lambda(n)}$$
$$=(1,f_1\exp(-t_2L)f_2\cdot\ldots\cdot\exp(-t_kL)f_k\cdot 1) \tag{3.25}$$

where 1 denotes the constant function 1 on S^{d-1}.

This is just the Feynman-Kac formula expressing that the continuum limit of the spin chain under consideration is represented by Brownian motion on S^{d-1}. This fact has been established for the one-dimensional Heisenberg model, i.e. for $f(\theta)=1-\cos\theta$, in [15]. The present calculations exhibit the universal character of the result in a simple fashion.

Let us also note that the region in the (β,λ) plane, where the two-point function $G_{\beta,\lambda}(0,x)$ is finite, looks as indicated in figure 1. Defining the susceptibility $\chi(\beta,\lambda)$ by

$$\chi(\beta,\lambda)=\int dx\,G_{\beta,\lambda}(0,x) \tag{3.26}$$

it follows from (3.2) that

$$\chi(\beta,\lambda)=\sum_{n=1}^{\infty}N_1(\beta)^{n-1}N_2(\lambda)^n. \tag{3.27}$$

Thus the critical line in figure 1 is determined by the equation

$$N_1(\beta)N_2(\lambda)=1 \tag{3.28}$$

and, of course, depends on the functions h and f.

The results derived above tell us that the continuum limit of $G_{\beta,\lambda}$, on the critical line, is a free propagator (or Brownian motion in $\mathbb{R}^d$), whereas in the limit $\lambda\to\infty$ and $\beta>0$ fixed we obtain Brownian motion on S^{d-1}, as far as the step correlation functions are concerned.

A simultaneous continuum limit of $G_{\beta,\lambda}(0,x)$ and of expectation values of functions of the step variables, i.e. of expectations of the form (3.8), can only exist at $(\beta,\lambda)=(0,\infty)$. In order to discuss this limit we define the masses $\mu(\beta,\lambda)$ and $m(\lambda)$ by†

$$\mu(\beta,\lambda)=-\lim_{|x|\to\infty}\frac{1}{|x|}\log G_{\beta,\lambda}(0,x) \tag{3.29}$$

and

$$m(\lambda)=-\lim_{n\to\infty}\frac{1}{n}\log\langle\hat r_0\cdot\hat r_n\rangle_\lambda \tag{3.30}$$

as in § 2. Then we require

$$\frac{\mu(\beta,\lambda)}{m(\lambda)}\to\text{constant}\qquad\text{as}\qquad(\beta,\lambda)\to(0,\infty) \tag{3.31}$$

which determines λ as a function of β.

Instead of elaborating on (3.31) we shall discuss the relation between the continuum limits of $G_{\beta,\lambda}$ and of tangent-tangent correlations in a more direct way.

Starting with (3.2) we use the fact that the Fourier transform $\tilde G_{\beta,\lambda}$ of $G_{\beta,\lambda}$ is given by

$$\tilde G_{\beta,\lambda}(q)=\sum_{n=1}^{\infty}\int \mathrm{d}r_1\ldots\mathrm{d}r_n\,P_0(r_1)\tilde P(r_1,r_2)\ldots\tilde P(r_{n-1},r_n)\exp[-\mathrm{i}q\cdot(r_1+\ldots+r_n)]. \tag{3.32}$$

Defining

$$F(q\cdot\hat r)=F_q(\hat r)=\frac{1}{N_1(\beta)}\int_0^\infty \mathrm{d}|r|\,|r|^{d-1}\exp[-\beta h(|r|)]\exp[-\mathrm{i}(q\cdot\hat r)\cdot|r|] \tag{3.33}$$

and taking $P_0(r)=N_1(\beta)^{-1}\exp[-\beta h(r)]$ (see the appendix), (3.32) can be rewritten as

$$\begin{aligned}\tilde G_{\beta,\lambda}(q)&=\sum_{n=1}^{\infty}N_1(\beta)^{n-1}N_2(\lambda)^n\int \mathrm{d}\Omega(\hat r_1)\ldots\mathrm{d}\Omega(\hat r_n)F(q\cdot\hat r_1)K_\lambda(\hat r_1,\hat r_2)\\&\qquad\times F(q\cdot\hat r_2)K_\lambda(\hat r_2,\hat r_3)\ldots F(q\cdot\hat r_{n-1})K_\lambda(\hat r_{n-1},\hat r_n)F(q\cdot\hat r_n)\\&=\sum_{n=1}^{\infty}N_1(\beta)^{n-1}N_2(\lambda)^n(1,F_q(K_\lambda F_q)^n1)\end{aligned} \tag{3.34}$$

where F_q is considered as a multiplication operator on $\mathcal{H}=L^2(S^{d-1},\mathrm{d}\Omega)$ and K_λ is defined by (3.6) and (3.9).

The continuum limit of $\tilde G_{\beta,\lambda}(q)$ is defined as

$$\tilde G(q)=\lim_{\theta\to\infty}\theta^{\eta-2}G_{\beta(\theta),\lambda(\theta)}(\theta^{-1}q) \tag{3.35}$$

† We assume here that h increases sufficiently rapidly so that the limit (3.29) is positive on the right of the critical line.

where $(\beta(\theta), \lambda(\theta))$ are chosen according to (3.31) and η is the anomalous scaling dimension, chosen such that the limit is finite.

We now note that

$$K^{\theta}_{\lambda(\theta)} \rightarrow \exp(-\lambda_R L) \tag{3.36}$$

if $\lambda(\theta)$ is chosen suitably (see (3.13)), and that

$$F_{q/\theta}(\hat{r})^{\theta} = F(\theta^{-1} q \cdot \hat{r})^{\theta} \rightarrow \exp(F'(0) q \cdot \hat{r}) \tag{3.37}$$

since $F(0) = 1$. Thus it follows from the Trotter product formula† that

$$(K_{\lambda(\theta)} F_{q/\theta})^{\theta} \rightarrow \exp(-\lambda_R L + F'(0) q \cdot \hat{r}) \tag{3.38}$$

where the function $q \cdot \hat{r}$ on S^{d-1} is regarded as a multiplication operator.

We now see from (3.35) that, if we set $n = \theta t$, and choose $\beta(\theta)$ such that

$$[N_1(\beta(\theta)) N_2(\lambda(\theta))]^{\theta} \rightarrow \exp(-\beta_R) \tag{3.39}$$

and take

$$\eta = 1 \tag{3.40}$$

then the sum in (3.34) turns into an integral over t and we get

$$\begin{aligned} \tilde{G}(q) &= \int_0^{\infty} \mathrm{d}t \exp(-\beta_R t)(1, \exp[-(\lambda_R L - \mathrm{i}cq \cdot \hat{r})t]1) \\ &= (1, (\lambda_R L - \mathrm{i}cq \cdot \hat{r} + \beta_R)^{-1} 1) \end{aligned} \tag{3.41}$$

where we have set $F'(0) = \mathrm{i}c$ $(c \in \mathbb{R})$.

In particular, we see from (3.41) that the continuum limit is independent of the function h, as expected, since the constant c can be absorbed into a redefinition of λ_R and β_R.

We emphasise that this expression for the propagator is different from the symmetrised continuum limit of the lattice propagator (2.42) (for any value of β_R, λ_R). One way to see this is to expand both expressions (2.42) and (3.41) in powers of q (for $\beta_R \neq 0$). This result is perhaps a little surprising, and is further discussed in the conclusion.

Remark. The construction leading to $\tilde{G}(q)$ above is applicable in a more general setting. Let M be a self-adjoint operator on $L^2(\mathbb{R}^d)$ such that the kernel $Q_t(r, r')$ of $\exp(-tM)$ is positive, and such that M has a positive ground-state wavefunction Ω, i.e.

$$\int Q_t(r, r')\Omega(r')\,\mathrm{d}r' = \Omega(r). \tag{3.42}$$

Then

$$R_t(r, r') = \Omega(r)^{-1} Q_t(r, r')\Omega(r') \tag{3.43}$$

is positive and can be interpreted as a transition probability since

$$\int R_t(r, r')\,\mathrm{d}r' = 1. \tag{3.44}$$

† In order for this formula to be applicable one has to make sure that the convergence in (3.36) is sufficiently rapid. We do not elaborate on this point here.

We now define a two-point function by

$$G_{\beta,\lambda}(r_1; x) = \sum_{n=1}^{\infty} \exp(-\beta n) \int R_\lambda(r_1, r_2) \ldots R_\lambda\left(r_{n-1}, x - \sum_1^{n-1} r_i\right) \mathrm{d}r_2 \ldots \mathrm{d}r_{n-1} \tag{3.45}$$

whose Fourier transform is

$$\begin{aligned}\tilde{G}_{\beta,\lambda}(r_1; q) &= \sum_{n=1}^{\infty} \exp(-\beta n) \int R_\lambda(r_1, r_2) \ldots R_\lambda(r_{n-1}, r_n) \exp\left(-\mathrm{i}q \cdot \sum_1^n r_i\right) \mathrm{d}r_2 \ldots \mathrm{d}r_n \\ &= \sum_{n=1}^{\infty} \exp(-\beta n) \int \Omega(r_1)^{-1} Q_\lambda(r_1, r_2) \ldots Q_\lambda(r_{n-1}, r_n) \\ &\quad \times \exp\left(-\mathrm{i}q \cdot \sum_1^n r_i\right) \Omega(r_n) \, \mathrm{d}r_2 \ldots \mathrm{d}r_n. \end{aligned} \tag{3.46}$$

Here r_1 is the initial velocity (or the first step).

The continuum limit is defined as in (3.35), and by applying the Trotter product formula one finds the continuum propagator

$$\begin{aligned}\tilde{G}(r_1; q) &= \int_0^\infty \mathrm{d}s \exp(-\beta_R s) \Omega(r_1)^{-1} \exp[-s(\lambda_R M + \mathrm{i}q \cdot r)](r_1, r') \Omega(r') \, \mathrm{d}r' \\ &= \int_0^\infty \mathrm{d}s \exp(-\beta_R s) \Omega(r_1)^{-1} (\delta_{r_1}, \exp[-s(\lambda_R M + \mathrm{i}q \cdot r)]\Omega) \\ &= \Omega(r_1)^{-1} (\delta_{r_1}, (\lambda_R M + \mathrm{i}q \cdot r + \beta_R)^{-1} \Omega) \end{aligned} \tag{3.47}$$

if β and λ are chosen as

$$\beta = \theta^{-1} \beta_R \qquad \lambda = \theta^{-1} \lambda_R. \tag{3.48}$$

As an illustration let us consider the ordinary Ornstein-Uhlenbeck process, for which we take

$$M = -\tfrac{1}{2}\Delta + \tfrac{1}{2}r^2 - \tfrac{1}{2}d \tag{3.49}$$

i.e. M is the Hamiltonian of the harmonic oscillator, and

$$\Omega(r) = \pi^{-d/4} \exp(-\tfrac{1}{2}r^2). \tag{3.50}$$

Observing that

$$M + \mathrm{i}q \cdot r = -\tfrac{1}{2}\Delta + \tfrac{1}{2}(r + \mathrm{i}q)^2 - \tfrac{1}{2}d + \tfrac{1}{2}q^2 \tag{3.51}$$

and that Δ is translation invariant we have

$$\exp[-t(M + \mathrm{i}q \cdot r)](r_1, r_2) = \exp(-\tfrac{1}{2}tq^2) Q_t(r_1 - \mathrm{i}q, r_2 - \mathrm{i}q). \tag{3.52}$$

Q_t is given by Mehler's formula:

$$\begin{aligned} Q_t(r_1, r_2) = \pi^{-d/2}[1 - \exp(-2t)]^{-d/2} \exp[\![-[1 - \exp(-2t)]^{-1} \\ \times \{(r_1^2 + r_2^2)\tfrac{1}{2}[1 + \exp(-2t)] - 2\exp(-t) r_1 \cdot r_2\}]\!] \end{aligned} \tag{3.53}$$

so

$$R_t(r_1, r_2) = \pi^{-d/2}[1 - \exp(-2t)]^{-d/2} \exp\{-[1 - \exp(-2t)]^{-1}[\exp(-t) r_1 - r_2]^2\}. \tag{3.54}$$

From (3.53) one gets that

$$Q_t(r_1 - \mathrm{i}q, r_2 - \mathrm{i}q) = Q_t(r_1, r_2) \exp\left(\frac{1 - \exp(-t)}{1 + \exp(-t)}[q^2 + \mathrm{i}q(r_1 + r_2)]\right). \tag{3.55}$$

Using this formula together with (3.52) in (3.47) one finds

$$\tilde{G}(r_1; q) = \int_0^\infty \mathrm{d}s \exp\left(-\beta_R s - \frac{q^2}{4\lambda_R^2}(2\lambda_R s - 3 + 4\exp(-\lambda_R s) - \exp(-2\lambda_R s)) + \frac{\mathrm{i}}{\lambda_R}(1 - \exp(-\lambda_R s)q \cdot r\right). \tag{3.56}$$

This result equals the Fourier transform of the standard form of the Ornstein-Uhlenbeck propagator [16].

Note, finally, that for all the processes constructed by this procedure, the anomalous dimension η is equal to 1.

4. Conclusion

The universality of the ordinary random walk is ensured by the central limit theorem or, phrased differently, the Markovian nature of the random walk. If we consider discretised random walks like the ones in §§ 2 and 3, the curvature-dependent interactions introduced there couple two neighbouring steps in such a way that the stochastic process is not Markovian in the step variables.

As expected, such short-range interactions do not change the critical behaviour of the random walks. For the piecewise linear model considered in § 3 this follows directly from a generalised central limit theorem (see the appendix), which allows us to reach the same conclusions as for the ordinary random walk. *Only* if we take the coupling constant of the curvature-dependent term in the action to infinity can we expect to get something qualitatively different from the ordinary random walk.

It is worthwhile rephrasing this in the language of the renormalisation group. It can be shown that the coupling constant $\alpha(\equiv \lambda^{-1})$ in front of the curvature-dependent term is asymptotically free [10, 13]. For the regularised theory this implies that $\alpha = 0$ ($\lambda = \infty$) is an ultraviolet stable fixed point. The only other fixed point on the critical line in the (β, λ) coupling constant plane (see figure 1) is the point $(\beta_0, \lambda = 0)$, corresponding to the critical point of the ordinary random walk. This point is infrared stable. Therefore, starting at a *finite* λ_c we will always be taken to this fixed point by the renormalisation group and the critical theory for finite λ_c is identical to one at $\lambda = 0$, as we indeed have verified by an explicit calculation. Only at $\lambda = \infty$ have we any chance of constructing a scaling limit corresponding to (1.1) with $\lambda\,(\text{continuum}) \neq 0$.

It is important to keep in mind that the continuum theory (defined by (1.1)) contains higher-derivative terms. Consequently, it is clear that the two-point function G, being the inverse of a third-order differential operator, cannot only be a function of the endpoints but must also have some reference to the initial value of the tangent, r_1, to the random walk.

There should not and does not exist a universal $G(x, y)$ which is model independent (after averaging over initial velocities), a fact which is exemplified by the three cases we have considered explicitly: the random walk on the lattice, the piecewise linear random walk and the Ornstein-Uhlenbeck velocity process.

In the lattice case, the distribution of tangents was chosen to be uniform among the $2d$ different ones (see (2.23)), since otherwise the propagator does not inherit the symmetry of the lattice. In the piecewise linear case considered in § 3, the distribution of r_1 was given by the product of the uniform distribution on the sphere and the weight function $\exp[-h(|r|)]$ (see (3.34)). In the Ornstein–Uhlenbeck case we chose the ground-state wavefunction of the generator of the process.

The difference between these three theories can be understood as follows. The first two processes have the common property that the length of subsequent steps are uncorrelated. This is a direct consequence of the scale invariance of the curvature-dependent part of the action. For the Ornstein–Uhlenbeck process and its generalisations the steps are correlated both in magnitude and direction.

The lattice walk can be regarded as a generalised one-dimensional Ising model, where the spin vectors can point in 2D different directions. The piecewise linear walks can be regarded as a classical one-dimensional spin chain and thus it should not be surprising that their scaling limits can be different, since the dynamical variables (i.e. the steps) have different symmetries. It is noteworthy that averaging over the rotation group does not make the lattice propagator equal to the one constructed from piecewise linear walks.

It is a most challenging problem to generalise the results of this paper to random surfaces. The first steps in that direction have been taken in [6], where it is shown that the phase diagram is similar to figure 1 and the extrinsic curvature coupling is asymptotically free. In the random surface theory the role of the mass of the tangent-tangent correlation is played by the string tension. In this case it is not clear whether the normal vectors to the surfaces have long-range correlations but it is tempting to conjecture that the vanishing of the string tension is equivalent to the divergence of the correlation length of normals. The role that a simple random walk plays in this paper is presumably taken over by branched polymers in the case of surfaces (see [7]).

Acknowledgment

One of us (BD) wishes to thank Jürg Fröhlich for discussions about subjects related to those treated in this paper.

Appendix

Let $X_1, X_2, \ldots,$ be random variables in $\mathbb{R}^d$ which are to be interpreted as the steps of a random walk starting at 0, such that the joint probability distribution of $X_1, \ldots, X_n$ is given by

$$P_n(x_1, \ldots, x_n) = P_0(x_1)P(x_1, x_2) \ldots P(x_{n-1}, x_n) \tag{A1}$$

and the conditional probability density $P(x, y)$ of a step y given by the previous step x is given by

$$P(x, y) = \frac{1}{N(\beta, \lambda)} \exp[-\beta h(|y|) - \lambda f(\theta(x, y))] \tag{A2}$$

where $\theta(x, y)$ denotes the angle between x and y, f is a continuous function on $[0, \pi]$, h is a function on $\mathbb{R}_+$ such that $\int |y|^2 \exp[-\beta h(|y|)]\, \mathrm{d}y < \infty$, $\forall \beta > 0$, and $N(\beta, \lambda)$ (which is independent of x) is determined by

$$\int P(x, y)\, \mathrm{d}y = 1. \tag{A3}$$

Furthermore, $P_0(x_1)$ is a fixed probability density of the first step. In the following we shall choose

$$P_0(x) = \left(\int \exp[-\beta h(|y|)]\, \mathrm{d}y \right)^{-1} \exp[-\beta h(|x|)] \tag{A4}$$

since the sequence $X_1, X_2, \ldots,$ is then stationary, but it is conceivable that the result stated below is independent of the choice of P_0.

We shall prove the following.

If λ is fixed, then the probability distribution of the variables

$$Z_n = \frac{X_1 + \ldots + X_n}{n} \tag{A5}$$

converges to a normal distribution

$$\mathcal{N}_A(z) = (2\pi)^{-d/2} (\det A)^{-1/2} \exp(-\tfrac{1}{2} zAz) \tag{A6}$$

for some positive $d \times d$ matrix A depending on β and λ in the sense that

$$\lim_{n\to\infty} P(Z_n < a) = \int_{Z<a} \mathcal{N}_A(z)\, \mathrm{d}z \tag{A7}$$

for all $a \in \mathbb{R}^d$, where $z < a$ means that each coordinate in z is less than the corresponding coordinate in a, and $P(Z_n < a)$ denotes the probability that $Z_n < a$, i.e.

$$P(Z_n < a) = \int_{(1/n)(x_1+\ldots+x_n)<a} P_n(x_1, \ldots, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

This result is a rather simple consequence of a local central limit theorem for weakly dependent random variables as formulated, e.g., in [17], theorem 18.5.1. That theorem is only formulated for real random variables, but, by considering the variables $a \cdot X_1, a \cdot X_2, \ldots,$ where $a \in \mathbb{R}^d$ is fixed, it follows easily that it also holds for vector valued random variables.

We show below that

$$\sigma_n^2 = \int (x_1 + \ldots + x_n)^2 P_n(x_1, \ldots, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \sim c_1 n + c_2 \tag{A8}$$

as $n \to \infty$, where c_1 and c_2 are constants. Furthermore, it is clear that

$$\int x_n^2 P_n(x_1, \ldots, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n < \infty. \tag{A9}$$

Knowing that (A8) and (A9) are fulfilled it is enough to prove that the sequence $X_1, X_2, \ldots,$ is uniformly mixing according to [17], i.e. we have to show that

$$\tau(m) \equiv \sup_{n,k,A,B} \left| \frac{P((X_1, \ldots, X_n) \in A, (X_{n+m+1}, \ldots, X_{n+m+k}) \in B)}{P((X_1, \ldots, X_n) \in A)} - \frac{P((X_1, \ldots, X_n) \in A) P((X_{n+m+1}, \ldots, X_{n+m+k}) \in B)}{P((X_1, \ldots, X_n) \in A)} \right| \tag{A10}$$

tends to 0 as $m \to \infty$. Here the supremum is taken over all $n, k \in \mathbb{N}$ and all non-zero measurable sets $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^k$, and

$$P((X_1, \ldots, X_n) \in A) = \int_{(x_1, \ldots, x_n) \in A} P_n(x_1, \ldots, x_n)\, dx_1 \ldots dx_n. \tag{A11}$$

In order to prove that $\tau(m) \to 0$ as $m \to \infty$, we shall rewrite (A10) a little. Let $\mathscr{H}_1 = L^2(\mathbb{R}^d, d\mu)$ be the Hilbert space of square integrable functions on $\mathbb{R}^d$ with respect to the measure

$$d\mu(x) = P_0(x)\, dx \tag{A12}$$

where P_0 is given by (4). Denote the inner product on $\mathscr{H}_1$ by $\langle \mid \rangle$ and let Q be the operator on $\mathscr{H}_1$, whose kernel with respect to $d\mu$ is

$$Q(x, y) = \frac{1}{M(\lambda)} \exp[-\lambda f(\theta(x, y))] \tag{A13}$$

where the normalisation constant $M(\lambda)$ is defined by

$$\int Q(x, y)\, dy = 1. \tag{A14}$$

Thus

$$(Qf)(x) = \int Q(x, y) f(y)\, d\mu(y) \tag{A15}$$

for $f \in \mathscr{H}_1$.

Considering $\mathscr{H}_1$ as $L^2(\mathbb{R}_+, d\nu) \otimes L^2(S^{d-1}, d\Omega)$ where

$$d\nu(r) = \left(\int_0^\infty r'^{d-1} \exp(-\beta h(r'))\, dr' \right)^{-1} r^{d-1} \exp(-\beta h(r))\, dr$$

and $d\Omega$ is the uniform measure on S^{d-1} considered in § 3, we notice that

$$Q = e \otimes K_\lambda$$

where e is the projection in $L^2(\mathbb{R}_+, d\nu)$ onto the constant functions and K_λ is the operator on $L^2(S^{d-1}, d\Omega)$ defined in § 3. In particular, it follows that the non-zero eigenvalues coincide with those of K_λ and the eigenfunctions are tensor products of $1 \in L^2(\mathbb{R}_+, d\nu)$ with the corresponding eigenfunctions of K_λ (spherical functions). Thus, when acting on those functions we may identify Q with K_λ.

It is then clear that

$$P_n(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = Q(x_1, x_2) \ldots Q(x_{n-1}, x_n)\, d\mu(x_1) \ldots d\mu(x_n). \tag{A16}$$

It is also obvious that Q is a symmetric operator on $\mathscr{H}_1$ with positive kernel and with norm $\leqslant 1$. Furthermore, the constant functions are eigenfunctions with eigenvalue 1. According to the Perron–Frobenius theorem 1 is then a non-degenerate eigenvalue and it is easy to see that -1 is not an eigenvalue. Thus, in particular, all eigenfunctions except the constants correspond to eigenvalues that are numerically less than 1.

We note that by rotational invariance (see § 3) the function

$$\phi(x) = \frac{1}{|x|} x \cdot x_0 \tag{A17}$$

is an eigenfunction for Q with eigenvalue $a<1$ for any $x_0\in\mathbb{R}^n\backslash\{0\}$. Thus it follows from (A16) that for $i\leqslant j\leqslant n$ we have

$$\begin{aligned}\int x_i\cdot x_jP_n(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\ &=\int x_i\cdot x_jQ(x_i,x_{i+1})\ldots Q(x_{j-1},x_j)\,\mathrm{d}\mu(x_i)\ldots\mathrm{d}\mu(x_j)\\ &=a^{|i-j|}\int x^2\,\mathrm{d}\mu(x)\end{aligned}\tag{A18}$$

from which (A8) easily follows.

Returning to (A10), it follows from (A16) that, if for given $A\subset\mathbb{R}^n$ and $B\subset\mathbb{R}^k$, we define

$$f(x)=\int_{(x_1,\ldots,x_{n-1},x)\in A}Q(x_1,x_2)\ldots Q(x_{n-1},x)\,\mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_{n-1})\tag{A19}$$

and

$$g(x)=\int_{(x,x_2,\ldots,x_k)\in B}Q(x,x_2)\ldots Q(x_{k-1},x_k)\,\mathrm{d}\mu(x_2)\ldots\mathrm{d}\mu(x_k)\tag{A20}$$

then f, $g\geqslant 0$ and

$$\langle f|1\rangle=\int f(x)\,\mathrm{d}\mu(x)=P((X_1,\ldots,X_n)\in A)\tag{A21}$$

$$\langle 1|g\rangle=\int g(x)\,\mathrm{d}\mu(x)=P((X_{n+m+1},\ldots,X_{n+m+k})\in B)\tag{A22}$$

and

$$\begin{aligned}\langle f|Q^mg\rangle&=\int f(x_n)Q(x_n,x_{n+1})\ldots Q(x_{n+m},x_{n+m+1})g(x_{n+m+1})\,\mathrm{d}\mu(x_n)\ldots\mathrm{d}\mu(x_{n+m+1})\\ &=P((X_1,\ldots,X_n)\in A,(X_{n+m+1},\ldots,X_{n+m+k})\in B).\end{aligned}\tag{A23}$$

Thus we have that

$$\tau(m)\leqslant\sup_{\substack{f,g\geqslant 0\\ \langle f|1\rangle,\langle 1|g\rangle\leqslant 1}}\frac{|\langle f|Q^mg\rangle-\langle f|1\rangle\langle 1|g\rangle|}{\langle f|1\rangle}\tag{A24}$$

where the supremum is over all non-negative functions f, $g\in\mathscr{H}_1$ whose integrals are $\leqslant 1$.

Since Q is rotationally invariant its non-zero eigenvalues $\lambda_0=1,\lambda_1,\lambda_2,\ldots,$ are degenerate with the same eigenfunctions and multiplicity d_l, $l=0,1,2,\ldots,$ as the eigenvalues of the Laplace-Beltrami operator on S^{d-1}. We let $\varphi_0=1$, $\varphi_{11},\ldots,\varphi_{1d_1}$, $\varphi_{21},\ldots,\varphi_{2d_2},\ldots,$ denote the corresponding normalised eigenfunctions. Then we have that (after ordering the non-zero eigenvalues suitably)

$$|\varphi_{li}|\leqslant cl^{\delta}\qquad l=1,2,\ldots;\ i=1,\ldots,d_l\tag{A25}$$

for some constants c, δ depending only on the dimension d (see, e.g., [14]). Moreover, the multiplicity d_l satisfies

$$d_l\sim c'l^{\delta'}\qquad\text{as}\qquad l\to\infty\tag{A26}$$

where c', δ' again only depend on d (see [14]) (and $\delta'>0$ if $d>2$).

Expanding $\langle f|Q^m g\rangle$ in terms of the eigenfunctions of Q, we now get

$$\begin{aligned}\frac{|\langle f|Q^m g\rangle-\langle f|1\rangle\langle 1|g\rangle|}{\langle f|1\rangle} &= \left|\sum_{l=1}^{\infty}\sum_{i=1}^{d_l}\lambda_l^m\frac{\langle f|\varphi_{li}\rangle\langle\varphi_{li}|g\rangle}{\langle f|1\rangle}\right| \\ &\leqslant \sum_{l=1}^{\infty}\sum_{i=1}^{d_l}|\lambda_l|^m(cl^\delta)^2 \\ &= c^2\sum_{l=1}^{\infty}d_l l^{2\delta}|\lambda_l|^m \end{aligned} \tag{A27}$$

for $f, g\geqslant 0$ and $\langle f|1\rangle, \langle g|1\rangle\leqslant 1$.

Since Q is clearly Hilbert–Schmidt we have that

$$\sum_{l=1}^{\infty}|\lambda_l|^2 d_l<\infty$$

and hence, by (A26),

$$|\lambda_l|\leqslant \text{constant}\times l^{-\delta'/2}. \tag{A28}$$

From (A24) and (A26)–(A28) it easily follows that $\tau(m)\to 0$ as $m\to\infty$, as desired, for $d>2$. For $d=1, 2$ we have $\delta=\delta'=0$ and the result also follows.

For $d=2$ a result similar to the one presented in this appendix was proven in [18].

References

[1] Hermanns J J and Ullman R 1952 *Physica* **18** 951
[2] Saito N, Takahashi K and Yunoki Y 1967 *J. Phys. Soc. Japan* **22** 219
[3] Polyakov A M 1986 *Nucl. Phys.* B **268** 406
[4] Ambjørn J and Durhuus B 1987 *Phys. Lett.* **118B** 253
[5] Durhuus B and Jonsson T 1986 *Phys. Lett.* **180B** 385
[6] Ambjørn J, Durhuus B and Jonsson T 1987 *Phys. Rev. Lett.* **58** 2619
[7] Durhuus B, Fröhlich J and Jonsson T 1984 *Nucl. Phys.* B **240** [FS12] 453
[8] Durhuus B, Fröhlich J and Jonsson T 1985 *Nucl. Phys.* B **257** 779
[9] Helfrich W 1985 *J. Physique* **46** 1263
Peliti L and Leibler S 1985 *Phys. Rev. Lett.* **54** 1690
Förster D 1986 *Phys. Lett.* **114A** 115
Kleinert H 1986 *Phys. Lett.* **174B** 335
David F 1986 *Europhys. Lett.* **2** 577
Olesen P and Yang S-K 1987 *Nucl. Phys.* B **283** 73
David F and Guitter E 1987 *Europhys. Lett.* **3** 1169
Pisarski R D 1986 *Phys. Rev. Lett.* **58** 1300; 1987 *Fermilab preprint* PUB-87/23-T
Curtright T L, Gandour G I, Thorn C B and Zachos C K 1986 *Phys. Rev. Lett.* **57** 799
Curtright T L, Gandour G I and Zachos C K 1986 *Phys. Rev.* D **34** 3811
Braaten E and Zachos C K 1987 *Phys. Rev.* D **35** 1512
[10] Pisarski R D 1986 *Phys. Rev.* D **34** 670
[11] Alonso F and Espriu D 1987 *Nucl. Phys.* B **283** 393
[12] Kleinert H 1986 *J. Math. Phys.* **27** 3003
[13] Ambjørn J, Durhuus B and Jonsson T 1987 *Europhys. Lett.* **3** 1059
[14] Helgason S 1984 *Groups and Geometric Analysis* (New York: Academic)
[15] Isaacson D 1982 *J. Math. Phys.* **23** 138
[16] Chandrasekhar S 1943 *Rev. Mod. Phys.* **15** 1
[17] Ibragimov I A and Linnik Yu V 1971 *Independent and Stationary Sequences of Random Variables* (Groningen: Walters-Nordhoff)
[18] Kadota T T 1983 *IEEE Trans. Inf. Theory* **IT-29** 241